Special Lagrangian submanifolds in the complex sphere

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Abstract

We construct a family of Lagrangian submanifolds in the complex shere whith a $SO(n)$-invariance property. Among them we find those which are special Lagrangian with respect with the Calabi-Yau structure defined by the Stenzel metric

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Introduction

Special Lagrangian submanifolds may be defined as those submanifolds which are both Lagrangian (an order 1 condition) and minimal (an order 2 condition). Alternatively, they are characterised as those submanifolds which are calibrated by a certain n-form (*cf* [HL]), so they have the remarkable property of being area minimizing. Their study have received many attention recently since connections with string theory have been discovered. More particularly, understanding fibrations of special Lagrangian (possibly with singularities) in Calabi-Yau manifolds of (complex) dimension 3 is crucial for mirror symmetry (*cf* [SYZ],[Jo]). Since the pioneering work of Harvey and Lawson [HL], where this notion were introduced, several authors ([Ha],[Jo],[CU2]) have discovered many families of special Lagrangian submanifolds in the complex Euclidean space, however very few examples (*cf* [Br]) of such submanifolds are known in other Calabi-Yau manifolds, where this notion is naturally extended. Maybe the main reason is that such manifolds are somewhat rare, in particular the existence of compact, Calabi-Yau manifolds is a hard result of S.-T. Yau [Y] involving non explicit solutions to some PDE. However

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there exists some intermediary examples of non-flat, non-compact Calabi-Yau manifolds, maybe the simplest of them being the complex sphere.

In this paper we describe a family of Lagrangian submanifolds of a complex variety of $\mathbb{C}^{n+1}$ with a $SO(n)$-invariance. In the case of the complex sphere, we obtain among them new examples of special Lagrangian submanifolds, which are a kind of generalization of the Lagrangian catenoid discovered in [HaLa] (*cf* also [CU1]).

In [CMU], minimal Lagrangian submanifolds have been studied in the complex hyperbolic and complex projective spaces respectively. This is somewhat close to our situation for the following reason: these spaces are not Calabi-Yau manifolds, and so these submanifolds are not minimizing *a priori* ; however Y.-G. Oh has proved in [Oh] that a minimal Lagrangian submanifold of the complex hyperbolic space is stable and without nullity, a property necessary for being a minimizer.

The paper is organized as follows:

In Section 1 we give a definition of Calabi-Yau manifolds, we state some basic facts on them and describe the Calabi-Yau structure yielded on the complex sphere by the Stenzel metric. In Section 2 we describe a general class of Lagrangian submanifolds whith some $SO(n)$-invariance immersed in some complex submanifold of $\mathbb{C}^{n+1}$. In the particular case of the complex sphere, we then check that they are also Lagrangian for the Calabi-Yau structure. In the last section we find in this class the special Lagrangian and describe them.

1 The Calabi-Yau structure on the complex sphere

We start with the following definitions — which are not the standard ones — proposed in [Jo]:

Definition 1 *Let $n \geq 2$. An almost Calabi-Yau manifold is a quadruple (X, J, ω, Ω) such that (X, J) is a n-dimensional complex manifold, ω the sym-*

plectic form of a Kähler metric g on X and Ω a non-vanishing holomorphic $(n,0)$-form.

(X, J, ω, Ω) is called a Calabi-Yau manifold if in addition

$$(1) \qquad \frac{\omega^n}{n!} = (-1)^{n(n-1)/2} \Omega \wedge \bar{\Omega},$$

where ω is the symplectic 2-form corresponding to g.

An important property of a Calabi-Yau manifold is that it is Ricci-flat, *i.e.* the Ricci curvature vanishes. Conversely, on a simply connected open subset a Ricci-flat Kähler metric yields a Calabi-Yau structure. The most simple example of Calabi-Yau manifold is $\mathbb{C}^n$ equipped with its standard structures, that is

$$\omega_0 = i \sum_j dz_j \wedge d\bar{z}_j$$

and

$$\Omega_0 = dz_1 \wedge \ldots \wedge dz_n.$$

Definition 2 *A submanifold L of real dimension n of X is said to be special Lagrangian if $\omega|_L \equiv \operatorname{Im} \Omega|_L \equiv 0$.*

The first condition says that L is Lagrangian and the second one that L is calibrated with respect to $\operatorname{Re} \Omega$. This implies that a special Lagrangian is always necessarily minimizing (*cf* [HL]).

We now describe a class of almost Calabi-Yau manifolds. Let P be a complex polynomial. Then the set

$$Q := \{(z_0, \ldots, z_n), P(z_0) = \sum_{j=1}^n z_j^2\}$$

is a complex submanifold of $\mathbb{C}^{n+1}$ which is smooth except if P admits double poles. For example, if $P(z) = z^2$, Q is made of two copies of $\mathbb{C}^n$ singularly connected at 0. Q inherits from $\mathbb{C}^{n+1}$ a Kähler structure.

On Q we define the following holomorphic $(n,0)$-form Ω, defined by

$$\Omega \wedge d(-P(z_0) + z_1^2 + \ldots + z_n^2) = dz_0 \wedge \ldots \wedge dz_n$$

We may give an explicit form on the set $\{P(z_0) \neq 0\}$:

$$\Omega = \frac{1}{\dot{P}(z_0)} dz_1 \wedge \ldots \wedge dz_n$$

In the remainder of the section we shall see that in the particular case of the complex sphere, that is when $P(z) = 1 - z^2$, Q is a Calabi-Yau manifold. This is a consequence of the following result due to Stenzel (cf [St]):

Proposition 1 *On the complex sphere $\{\sum_{j=0}^n z_j^2 = 1\}$, there exists a Ricci-flat metric, whose corresponding symplectic form is*

$$\omega_{St} = i\partial\bar{\partial}u(r^2) = i\sum_{j,k} \frac{\partial^2}{\partial z_j \partial \bar{z}_k} u(r^2) dz_j \wedge d\bar{z}_k$$

where $r^2 = \sum_{j=0}^n z_j \bar{z}_j$ and u is some smooth real function.

In dimension 2, we have an explicit form: $u(r^2) = \sqrt{1 + r^2}$, and in higher dimension, u is defined as the solution of some differential equation. As the complex sphere is simply connected, this result shows that it is Calabi-Yau.

It remains to show that the holomorphic n-form Ω that we have introduced is the one of the Calabi-Yau structure, up to a multiplicative constant. So see this, let f be the ratio of these two holomorphic forms. We observe that Ω is invariant under the natural action of $SO(n+1)$ defined by $z = x + iy \mapsto Az = Ax + iAy, \forall A \in SO(n+1)$. But this is also the case for the Stenzel metric, and thus for the corresponding holomorphic n-form. This implies that f is constant on the orbits of the action of $SO(n+1)$. But generic orbits have codimension 1 in the complex sphere, so the image of f in $\mathbb{C}$ has at most real dimension 1 locally, which forces f to be constant, as it is holomorphic. Thus the two forms are proportional.

2   A Lagrangian Ansatz

Let γ be a curve into $\mathbb{C}$. Then

$$X: \begin{array}{ccc} I \times \mathbb{S}^{n-1} & \to & Q \\ (s,x) & \mapsto & (\gamma(s), \sqrt{P(\gamma(s))}x_1, \ldots, \sqrt{P(\gamma(s))}x_n), \end{array}$$

where $(x_1, \ldots, x_n) = x$, is a an Lagrangian immersion for ω_0. In spite of the indetermination of the complex square root, the immersion is well defined

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because of the invariance of $\mathbb{S}^{n-1}$ by $x \mapsto -x$. However it becomes singular when γ is a zero of P.

From now on we consider only the case $P(\gamma) = 1 - \gamma^2$, that is the complex sphere equipped with the Stenzel form

Due to the larger symmetry of the complex sphere, we may slightly generalise the previous ansatz: we embed $\mathbb{S}^{n-1}$ into $\mathbb{S}^n$ in the following way: $\mathbb{S}^{n-1} = \mathbb{S}^n \cap \{x_0 = 0\}$. For some $p \in \mathbb{S}^n$, we shall note A_p any element of $SO(n+1)$ such that $A_p(\{x_0 = 0\}) = p^{\perp}$. Let γ be a curve into $\mathbb{C}^*$. Then we define

$$
\begin{array}{rcc}
X_{p,\gamma} : & I \times \mathbb{S}^{n-1} & \to & Q \\
& (s, x) & \mapsto & \gamma(s)p + \sqrt{1 - \gamma(s)^2} A_p x,
\end{array}
$$

and it is clear that the image does not depend on the choice of A_p.

Lemma 1 X is also Lagrangian for the Stenzel metric.

Proof. Let $(z_0, \ldots, z_n)$ be coordinates on $\mathbb{C}^n$ such that $p = (1, 0, \ldots, 0)$. As long as γ does not vanish we can use $(z_1, \ldots, z_n)$ as coordinates on a neighbourhood of the image of X in Q. In particular, we have

$$
\frac{\partial z_0}{\partial z_j} = -\frac{z_j}{z_0}.
$$

We compute that

$$
\frac{\partial}{\partial z_j} r^2 = \bar{z}_j - \frac{\bar{z}_0}{z_0} z_j
$$

$$
\frac{\partial^2}{\partial z_j \partial \bar{z}_k} r^2 = \delta_{jk} + \frac{z_j \bar{z}_k}{|z_0|^2}
$$

So we deduce that

$$
\omega_{St} = i \sum_{j,k}^n a_{jk} dz_j \wedge d\bar{z}_k
$$

with

$$
a_{jk} = \left(\delta_{jk} + \frac{z_j \bar{z}_k}{|z_0|^2} \right) u' + 2\mathrm{Re}\left(\bar{z}_j z_k - \frac{\bar{z}_0}{z_0} z_j z_k \right) u''
$$

Hence we can decompose ω_{St} as

$$
\omega_{St} = u' \omega_0 + \omega_1,
$$

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where

$$\omega_1 := i \sum_{j,k}^n \left(\frac{z_j \bar{z}_k}{|z_0|^2} u' + 2\text{Re} \left(\bar{z}_j z_k - \frac{\bar{z}_0}{z_0} z_j z_k \right) u'' \right) dz_j \wedge d\bar{z}_k$$

As X is Lagrangian for ω_0, it remains to show that it is also Lagrangian for ω_1. This will be a consequence of the fact that the two following 2-forms vanish on X :

$$\sum_{j,k}^n z_j \bar{z}_k dz_j \wedge d\bar{z}_k$$

$$\sum_{j,k}^n z_j z_k dz_j \wedge d\bar{z}_k$$

We shall note $T_\alpha = (X_\alpha^1, \ldots, X_\alpha^n)$, $1 \le \alpha \le n-1$, a basis of tangent vectors to $\mathbb{S}^{n-1}$ at $x = (x_1, \ldots, x_n)$.

This yields a basis of the tangent space at $X(s, x)$:

$$X_s = (\dot{\gamma}, \frac{\dot{\gamma}\dot{P}(\gamma)}{2\sqrt{P(\gamma)}} x_1, \ldots, \frac{\dot{\gamma}\dot{P}(\gamma)}{2\sqrt{P(\gamma)}} x_n)$$

$$X_\alpha = (0, \sqrt{P(\gamma)} X_\alpha^1, \ldots, \sqrt{P(\gamma)} X_\alpha^n).$$

Now we have:

$$\left(\sum_{j,k} \bar{z}_j z_k dz_j \wedge d\bar{z}_k \right) (X_s, X_\alpha) = \sum_{j,k} |P(\gamma)| x_j x_k \begin{vmatrix} \frac{\dot{\gamma}\dot{P}(\gamma)}{2\sqrt{P(\gamma)}} x_j & \sqrt{P(\gamma)} X_j^\alpha \\ \frac{\dot{\gamma}\dot{P}(\gamma)}{2\sqrt{P(\gamma)}} x_k & \overline{\sqrt{P(\gamma)} X_k^\alpha} \end{vmatrix}$$

$$= \sum_{j,k} |P(\gamma)| x_j x_k \left(\frac{\dot{\gamma}\dot{P}(\gamma)}{2\sqrt{P(\gamma)}} \overline{\sqrt{P(\gamma)}} x_j X_\alpha^k - \overline{\frac{\dot{\gamma}\dot{P}(\gamma)}{2\sqrt{P(\gamma)}}} \sqrt{P(\gamma)} x_k X_\alpha^j \right)$$

$$= |P(\gamma)| \frac{\dot{\gamma}\dot{P}(\gamma)}{2\sqrt{P(\gamma)}} \overline{\sqrt{P(\gamma)}} \sum_j x_j^2 \sum_k x_k X_\alpha^k - |P(\gamma)| \overline{\frac{\dot{\gamma}\dot{P}(\gamma)}{2\sqrt{P(\gamma)}}} \sqrt{P(\gamma)} \sum_k x_k^2 \sum_j x_j X_\alpha^j$$

The latter vanish because tangent vectors to the real sphere at some point are orthogonal to this point.

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$$\left(\sum_{j,k} z_j \bar{z}_k dz_j \wedge d\bar{z}_k\right)(X_\alpha, X_\beta) = \sum_{j,k} |P(\gamma)| x_j x_k \begin{vmatrix} \sqrt{P(\gamma)} X_j^\alpha & \sqrt{P(\gamma)} X_j^\beta \\ \overline{\sqrt{P(\gamma)}} X_k^\alpha & \overline{\sqrt{P(\gamma)}} X_k^\beta \end{vmatrix}$$

$$= \sum_{j,k} |P(\gamma)| x_j x_k \left(P(\gamma) X_j^\alpha X_k^\beta - P(\gamma) X_k^\alpha X_j^\beta\right) = 0$$

The computations for showing that also $\sum_{j,k} z_j \bar{z}_k dz_j \wedge dz_k$ vanishes on X are analogous.

3 Special Lagrangian submanifolds

Into the class of Lagrangian defined in the previous section, and in the two cases where we have a Calabi-Yau structure, we shall look for those calibrated by $\mathrm{Re}\,\Omega$, that is those on which $\mathrm{Im}\,\Omega$ vanishes.

3.1 Flat case of $P(z) = z^2$

The holomorphic $(n,0)$-form we use is the standard one

$$\Omega_0 = dz_1 \wedge \ldots \wedge dz_n$$

The equation is $\mathrm{Im}\,(\dot{\gamma}\gamma^{n-1}) = 0$, which is easily integrated as $\mathrm{Im}\,(\gamma^n) = C$, where C is some real constant. It turns that we have two types of solutions:

- If $C = 0$, γ is the union of n lines passing trough the origin, with angles $\pi/n[2\pi]$ and the corresponding special Lagrangian are just Lagrangian n-spaces,

- If $C \neq 0$, the curve is asymptotic to two of the lines described above. All the curves (and so the corresponding Lagrangian as well) are congruent. We recognize the Lagrangian catenoid which was first identified in [HL] and characterised in [CU1] as the only special Lagrangian of $\mathbb{C}^n$ which is foliated by $(n-1)$-dimensional round spheres.

3.2 Case of the complex sphere $P(z) = 1 - z^2$

Before starting the computations, we observe that we already know a special Lagrangian in the complex sphere: if a Calabi-Yau has a real structure which is in some sense compatible (*cf* [Au], pp. 62–64), then the set of real points, if non empty, is a special Lagrangian submanifold. In our case, real points constitute an real sphere embedded in Q, so this will be a trivial solution to our problem.

We now compute the holomorphic form on X :

$$\Omega(X_s, X_1, \ldots, X_{n-1}) = \frac{1}{\dot{P}(\gamma)} \begin{vmatrix} \frac{\dot{\gamma}\dot{P}(\gamma)}{2\sqrt{P(\gamma)}}x_1 & \sqrt{P}X_1^1 & \cdots & \sqrt{P}X_{n-1}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\dot{\gamma}\dot{P}(\gamma)}{2\sqrt{P(\gamma)}}x_n & \sqrt{P}X_1^n & \cdots & \sqrt{P}X_{n-1}^n \end{vmatrix}$$

$$= \frac{\dot{\gamma}}{2\sqrt{P(\gamma)}}\sqrt{P(\gamma)}^{n-1} \begin{vmatrix} x_1 & X_1^1 & \cdots & X_{n-1}^1 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & X_1^n & \cdots & X_{n-1}^n \end{vmatrix}$$

$$= \dot{\gamma}\sqrt{P(\gamma)}^{n-2} C_{\mathbb{R}},$$

where $C_{\mathbb{R}}$ is some real constant.

Thus the differential equation for γ is

$$\mathrm{Im}\left(\dot{\gamma}\sqrt{1-\gamma^2}^{n-2}\right) = 0.$$

This equation has two singular points ± 1 and is regular elsewhere. Moreover, there is always a simple solution to this equation, the real segment $[-1, 1]$. This corresponds to the standard embedding of the real sphere. We shall analyse separately the even and odd cases.

3.2.1 Even case

In this case, the equation is polynomial and it is easy to find a first integral: we integrate $\mathrm{Im}\left(\dot{\gamma}(1-\gamma^2)^{n/2-1}\right) = 0$ and we get $\mathrm{Im}\left(Q(\gamma)\right) = C$, where Q is some polynomial such that $Q'(z) = (1-z^2)^{n/2-1}$. The degree of Q is $n-1$ and the the integral curves are algebraic curves of degree $n-1$ as well.

In order to have a better description of the solutions, we perform an asymptotic analysis of the equation when $|\gamma| \to \infty$ and when $\gamma \sim \pm 1$.

If $|\gamma| >> 1$, we have

$$\text{Im}\left(\dot{\gamma}(1-\gamma^2)^{n/2-1}\right) \sim -\text{Im}\left(\dot{\gamma}\gamma^{n-2}\right),$$

so asymptotically, the phase portrait of the equation looks like the one of the flat case (however not of the same dimension). In particular all integral curves are asymptotic to the half lines $\{\arg(z) = k\pi/(n-1)\}$, $1 \le k \le 2n-2$.

Next we write $\gamma = 1 + y$ and we find that for y close to 0,

$$\text{Im}\left(\dot{\gamma}(1-\gamma^2)^{n/2-1}\right) \sim \text{Im}\left(\dot{y}(-2y)^{n/2-1}\right)$$

So close to the singular point 1, the integral lines look like the level sets of

$$\text{Im}\left(-1/n(-2y)^{n/2}\right) = C.$$

In particular, the level set of level 0 is the union of n branches asymptotic in 1 to the half-lines $\{\arg(z) = 2k\pi/n + \pi\}$, $1 \le k \le n$, one of them being of course the real segment $[-1, 1]$.

At the point -1 the situation is exactly symmetric with respect to the reflexion $z \to -\bar{z}$.

3.2.2 Odd case

When n is odd, there is also a first integral which takes the following form

$$\text{Im}\left(\gamma\sqrt{1-\gamma^2}R(\gamma) + A\arcsin(\gamma)\right),$$

where R is a polynomial of degree $n-3$ and A a real constant. This quantity is not well-defined on the whole complex plane, however the level set lines can be defined locally.

If $|\gamma| >> 1$, we have

$$\text{Im}\left(\dot{\gamma}\sqrt{1-\gamma^2}^{n-2}\right) = \text{Im}\left(\dot{\gamma}\sqrt{1+(i\gamma)^2}^{n-2}\right) \sim \text{Im}\left(\dot{\gamma}(i\gamma)^{n-2}\right) = \pm\text{Im}\left(i\dot{\gamma}\gamma^{n-2}\right)$$

Again, the phase portrait looks asymptotically as in the flat case, but this time up to a rotation of angle $\pm\pi/2(n-1)$. In particular, all curves are asymptotic to the half lines $\{\arg(z) = k\pi/(n-1) + \pi/2(n-1)\}$, $1 \le k \le 2n-2$.

The asymptotic analysis next to the singular points ± 1 is the same than in the even case.

3.2.3 Conclusion

From these remarks, we can describe the general picture of the phase portrait :

- There is a singular curve made of the real segment $[-1,1]$ and of $2n-2$ branches, one half of them starting from 1 and the other half from -1, making between them an angle of $2\pi/n$ and going to infinity where they are asymptotic to the half lines $\{\arg(z)=k\pi/(n-1)\}$, $1\leq k\leq 2n-2$ when n is even and to $\{\arg(z)=k\pi/(n-1)+\pi/2(n-1)\}$, $1\leq k\leq 2n-2$ when n is odd.

- the other curves are smooth and have two ends asymptotic to two successive branches described above

For the corresponding special Lagrangian in Q, we deduce the following:

- The standard embedding of the real sphere,

- $2n-2$ special Lagrangian, touching the real sphere at the north pole $(1,0,\ldots,0)$ for one half of them and at the south pole $(-1,0\ldots,0)$ for the other half.

- A one-parameter smooth families of special Lagrangian with two ends asymptotic to two members of the above family.

Remark 1 *In the case of dimension* 2, *the integral lines are simply the horizontal lines.*

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